Social orderings for the assignment of indivisible objects

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Abstract

In the assignment problem of indivisible objects with money, we study social ordering functions which satisfy the requirement that social orderings should be independent of changes in preferences over infeasible bundles. We combine this axiom with efficiency, consistency and equity axioms. Our result is that the only social ordering function satisfying those axioms is the leximin function in money utility.

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1 Introduction

Consider that a group of agents have to assign objects among them. All objects are desirable but each agent may consume at most one object. Let us think of appartments in a housing complex, seats at a concert, parking lots, tasks in a board of directors, etc. Since the value of the objects may differ considerably among agents and between objects, monetary compensations are allowed for those who do not receive any objects or who receive an object

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they deem of low value. What could be an equitable way of assigning objects among those agents, and how should the compensations be computed?

The standard approach to this problem consists in looking for allocation rules (see Thomson [22] for a survey). An allocation rule specifies which feasible assignments are the most desirable as a function of the parameters of the problem, that is, the set of agents, the set of objects, and the agents' preferences over the objects. An alternative approach consists in looking for social ordering functions. A social ordering function specifies a complete ranking of the feasible allocations as a function of the parameters.

This paper deals with such social ordering functions. In many circumstances, indeed, it is not sufficient to know which assignments are the most desirable. The main reason comes from information and incentive constraints, which often prevents the planner from reaching first best alternatives. Given agents' incentive to misreveal their preferences, given the information they own about each other, the set of reachable alternatives may have very different shapes, and typically does not contain the most desirable alternatives. Having defined a social ordering allows the planner to solve the normative problem in all those cases, by simply maximizing the ordering under the relevant incentive constraints.[1]

Several recent studies of fairness in economic domains have succeeded in building social ordering functions based on fairness properties (see Fleurbaey [6], Fleurbaey and Maniquet [7], Fleurbaey and Maniquet [8] and Maniquet and Sprumont [12]). That undertaking is clearly related to the social choice tradition of defining social welfare in such models. The social choice literature, however, has focussed on properties of independence with respect to changes in preferences, and it has mainly uncovered impossibility results (see Le Breton and Weymark [11] for a survey). In the recent literature on social ordering functions, independence properties are weakened and the emphasis shifts towards fairness properties.

This paper is at the intersection of those two branches of literature. Indeed, we combine an independence property borrowed from social choice literature with consistency and fairness properties borrowed from fair social ordering literature and still obtain a possibility result (moreover, we characterize a unique social ordering function). The independence property we

[1]Incentives are not the only reason why first best allocations may not be available. For instance, there may exist a status quo and reallocating objects may be too costly, so that only changes in monetary compensations are possible. Or there may be other feasibility constraints.

study requires that the social orderings be independent of preferences over infeasible alternatives (cfr. Plott [16]; see Le Breton and Weymark [11] for a survey on applications of the Plott axiom in economic domains). In our setting, this amounts to requiring that preferences over objects which are not available should not matter, which seems extremely natural.

The first, intermediary, result we present in this paper offers us a characterization result of the social ordering functions which satisfy our independence property together with efficiency and consistency requirements. This result makes clear why traditional social choice theory and the fair allocation literature can be reconciled in the assignment of indivisible objects model. Indeed, the key feature of this model turns out to be that it is always possible not to assign any object to an agent (while, maybe, compensating her for not receiving anything). Therefore, there exists a non-degenerate part of agents' consumption set their preferences over which always matter. The result says the following: when combined with efficiency and consistency requirements, the independence property forces us to focus on that part of the consumption sets, that is, on what can be called the money utilities. It turns out that money utilities information are sufficient to build complete rankings of feasible allocations, whatever those allocations are.

Our second and main result shows that it is possible to use fairness properties to select among all possible rankings based on money utilities. But it turns out that not all degree of inequality aversion is finally allowed in this setting. Indeed, the only social ordering function which is compatible with an anonymity axiom and an appropriate version of the Pigou-Dalton principle of transfer[2] turns out to be the money utility leximin function (according to the leximin, a vector is preferred to another one if the smallest element of the former is larger than that of the latter, or, in case of a tie, if the second smallest element is larger, and so on).

One core issue in welfare economics has traditionnally been the design of social objectives suitable for applications in resource allocation (or reallocation) problems. By mainly uncovering impossibilities, the theory of social choice in economic environment has failed to offer suitable objectives. By focussing on allocation rules, the theory of fair allocation has offered objectives which are difficult to use in applications. By combining those approaches,

[2]This principle says that a money transfer from a richer agent to a poorer one decreases inequality. In the-one dimensional income inequality measurement framework, this principle is compatible with any degree of inequality aversion (see, e.g., Chakravarty [4]).

this paper proposes a way out of the dilemma. Whether or not the new approach can be applied to other problems of interest to public economics is the question addressed by the recent studies of social ordering functions, a question which remains largely open.

The paper is organized as follows. We define the model in Section 2. We define our main axiom, *Independence of Preferences over Infeasible Bundles*, as well as other axioms in Section 3, where we also develop and discuss our first result. Then, we define the fairness axioms in Section 4 and prove our main result. We conclude the paper in Section 5.

2 The model

Let us begin with an example. A university department has a set of housings on campus to allocate to visitors. There are more visitors than housings, and no rights have been a priori allocated to visitors. Housings have to be allocated, and visitors who are not assigned campus housing should be given a compensation or a subsidy to find a housing elsewhere. All visitors would prefer to be located on campus. The process of allocating housings should depend on visitors' preferences over them. If this process consists of defining a complete ranking on possible ways of allocating housings and organizing transfers among visitors, then the department will be able to deal with feasibility constraints such as visitor i was already present last year and should not be asked to move, visitor j suffers from allergies and should not be located too close to visitor k who has a pet, housing a need repair and is no longer available, etc... Moreover and most importantly, when the assignment process is chosen, if it turns out that visitors are likely to manipulate information regarding their preferences, then the department has to design a revelation mechanism and may choose among the available mechanisms the one that is likely to yield the highest possible assignment in the ranking.

There is an infinite set $\mathcal{A}$ of objects, and an infinite set $\mathcal{N}$ of agents. In specific economies, agents may be assigned either an available object from $\mathcal{A}$ or no object at all. In the latter case, we say that this agent receives the "null object", which is denoted ν. Let $\mathcal{A}^* \equiv \mathcal{A} \cup \{\nu\}$. An *economy* is a list $E = (N, A, R)$ consisting of a finite set of agents $N \subset \mathcal{N}$, a finite set of objects $A \subset \mathcal{A}$, and a finite list of preferences $R = (R_i)_{i \in N}$ defined on $\mathcal{A}^* \times \mathbb{R}$. We assume that there are at least two agents and that there is at least one object to assign but never more objects than agents, that is, $\#N \geq 2$ and $1 \leq \#A \leq \#N$. Let I_i be the indifference relation associated to R_i, and P_i the strict preference relation. We assume that preferences are continuous and strictly monotone with respect to money. We also assume that all objects are desirable and their values is always finite, that is, for all $(a_i, m_i) \in \mathcal{A} \times \mathbb{R}$, $(a_i, m_i)\ P_i\ (\nu, m_i)$, and there exists $m_i' \in \mathbb{R}$, such that $(a_i, m_i)\ I_i\ (\nu, m_i')$. Let $\mathcal{R}$ denote the set of all such preference relations. Let $\mathcal{E}$ denote the set of all such economies.

A *bundle for an agent* $i \in N$ *in an economy* $E = (N, A, R) \in \mathcal{E}$ is a pair $z_i = (a_i, m_i) \in A \cup \{\nu\} \times \mathbb{R}$. A *feasible allocation for an economy* $E = (N, A, R) \in \mathcal{E}$ is a list $z = (z_i)_{i \in N}$ such that no two agents are assigned the same "real" object, all assignments are from the set of available objects, and the sum of monetary compensations does not exceed 0. Let $Z(E)$ denote the set of feasible allocations for E, that is, $z = ((a_i, m_i))_{i \in N} \in Z(E)$ if and only if

$$\begin{aligned} a_j = a_k &\Rightarrow a_j = \nu = a_k, \\ \cup_{i \in N} a_i &\subseteq A \cup \{\nu\}, \\ \sum_{i \in N} m_i &\leq 0. \end{aligned}$$

Feasible allocations are thus defined in a way which allows us to assign only a subset of the available objects. We come back to this assumption in the conclusion. For an economy $E = (N, A, R) \in \mathcal{E}$, for a feasible allocation $z \in Z(E)$, for $M \subset N$, we write R_M to denote the restriction of R to members of M, and z_M to denote the similar restriction of z.

A *social ordering* for an economy $E = (N, A, R) \in \mathcal{E}$ is a complete, reflexive and transitive ranking of the feasible allocations. A *social ordering function* $\mathbf{R}$ associates to each economy $E \in \mathcal{E}$ a social ordering $\mathbf{R}(E)$ for this economy. Let $\mathbf{I}$ be the indifference relation associated to $\mathbf{R}$, and $\mathbf{P}$ the strict preference relation.

3 Independence axiom and the money utility property

In order to define social ordering functions, we impose axioms capturing some desirable property for such functions. Our main axiom in this paper is borrowed from the (Arrovian) social choice theory on economic domains. It was introduced by Plott [16] in the abstract framework, and later applied to economic domains (see Le Breton and Weymark [11]). We call it *Independence of Preferences over Infeasible Bundles*. It requires that changes in agents preferences which leave unaffected their preferences over feasible bundles should not affect the social ordering. Formally,

Independence of Preferences over Infeasible Bundles For all $E = (N, A, R)$, $E' = (N, A, R') \in \mathcal{E}$, if for all $i \in N, a, a' \in A \cup \{\nu\}, m, m' \in \mathbb{R}$,

$$(a, m) \ R_i \ (a', m') \Leftrightarrow (a, m) \ R'_i \ (a', m'),$$

then for all $z, z' \in Z(E)$,

$$z \, \mathbf{R}(E) \, z' \Leftrightarrow z \, \mathbf{R}(E') \, z'.$$

Independence of Preferences over Infeasible Bundles can only have consequences over the social ordering function if the set of available objects is allowed to vary. The following axiom is inspired by the Consistency property which has been extensively studied in the fair allocation literature (see Thomson [21]). Consistency usually applies to allocation rules. It requires that if an allocation is selected for an economy, then the suballocation obtained by removing a subset of agents with their assignments of good and money be also selected for the subeconomy (see Tadenuma and Thomson [19]). Adapting this property in our setting, we obtain the following *Consistency* axiom. If an allocation is socially as good as another, and if a subset of agents are assigned exactly the same bundles in both allocations, then removing those agents with their assignments should not change the social preference, that is, the suballocation obtained from the first allocation by removing those agents should still be as good for the subeconomy as the suballocation obtained from the second allocation, provided those suballocations are feasible in the subeconomy. Formally,

Consistency For all $E = (N, A, R) \in \mathcal{E}, M \subset N, z, z' \in Z(E)$, if $z_i = z'_i$ for all $i \in N \setminus M$, then

$$\begin{aligned} z \, \mathbf{R}(E) \, z' \quad &\Rightarrow \quad z_M \, \mathbf{R}(M, A \setminus \cup_{i \in N \setminus M} \{a_i\}, R_M) \, z'_M, \text{ and} \\ z \, \mathbf{P}(E) \, z' \quad &\Rightarrow \quad z_M \, \mathbf{P}(M, A \setminus \cup_{i \in N \setminus M} \{a_i\}, R_M) \, z'_M, \end{aligned}$$

provided $z_M, z'_M \in Z\left(M, A \setminus \cup_{i \in N \setminus M} \{a_i\}, R_M\right)$.

Nothing prevents us up to now from assigning the objects independently of preferences. We therefore impose Paretian type axioms. In this paper we stick to the traditional *Strong Pareto* requirement and its weak consequence of *Pareto Indifference*. *Strong Pareto* requires that if each agent weakly prefers her assigned bundle in one allocation over that in another, then the former allocation be also weakly preferred by the society to the latter. If, in addition, at least one agent strictly prefers the former allocation, then it is also strictly preferred. *Strong Pareto* will only be used in the next Section, but we define it here, due to its logical relationship with *Pareto Indifference*, which will play a crucial role in our first result.

Strong Pareto For all $E = (N, A, R) \in \mathcal{E}$, $z, z' \in Z(E)$, if $z_i \, R_i \, z'_i$ for all $i \in N$, then $z \, \mathbf{R}(E) \, z'$. If, in addition, $z_j \, P_j \, z'_j$ for some $j \in N$, then $z \, \mathbf{P}(E) \, z'$.

Pareto Indifference requires that two allocations which are deemed equally good by all agents be also viewed socially equivalent.

Pareto Indifference For all $E = (N, A, R) \in \mathcal{E}$, $z, z' \in Z(E)$, if $z_i \, I_i \, z'_i$ for all $i \in N$, then $z \, \mathbf{I}(E) \, z'$.

In our first result, we characterize the class of social welfare functions which satisfy our main axiom, *Independence of Preferences over Infeasible Bundles,* together with *Consistency* and *Pareto Indifference*. If a social ordering function satisfies those three axioms, then it also satisfies the property that in order to evaluate two allocations, only the quantity of money leaving the agents indifferent between their assigned bundles and not receiving any object matters. This quantity of money is called the money utility of the corresponding bundle. Money utilities allow us to construct numerical representation of each preference relation. That only money utility matters comes from the fact that, since it is always possible to assign the null object to an agent, the part of the consumption set where money utilities are computed is always part of the set of feasible bundles, and, more precisely, is the proper intersection of all the possible sets of feasible bundles. By the assumption that the values of the objects are always finite, it turns out that the information given by the money utilities is all what matters for the construction of social ordering functions.

Money Utility Property For all $E = (N, A, R), E' = (N, A', R') \in \mathcal{E}$, for all $z, z' \in Z(E), y, y' \in Z(E')$, if there exist $m^*, m^{*\prime} \in \mathbb{R}^N$ such that

$$z_i \, I_i \, (\nu, m_i^*) \, I'_i \, y_i \text{ and } z'_i \, I_i \, (\nu, m_i^{*\prime}) \, I'_i \, y'_i$$

then

$$z\,\mathbf{R}(E)\,z' \Leftrightarrow y\,\mathbf{R}(E')\,y'$$

Lemma 1 *If a social ordering function* **R** *satisfies* Independence of Preferences over Infeasible Bundles, Consistency*, and* Pareto Indifference*, then it satisfies the* Money Utility Property*.*

Proof. We begin with the following important claim.

Claim: Let $E = (N, A, R) \in \mathcal{E},\ z, z' \in Z(E)$. Let $\widetilde{E} = \left(\widetilde{N}, \widetilde{A}, \widetilde{R}\right) \in \mathcal{E}$ be such that $N \cap \widetilde{N} = \emptyset$ and $A \cap \widetilde{A} = \emptyset$. Observe that $\left(E, \widetilde{E}\right) = \left(N \cup \widetilde{N}, A \cup \widetilde{A}, \left(R, \widetilde{R}\right)\right) \in \mathcal{E}$. Let $\widetilde{z} \in Z(\widetilde{E})$. If **R** satisfies *Consistency*, then

$$\begin{aligned} z\,\mathbf{R}(E)\,z' &\Rightarrow (z, \widetilde{z})\,\mathbf{R}(E, \widetilde{E})\,(z', \widetilde{z}), \text{ and} \\ z\,\mathbf{P}(E)\,z' &\Rightarrow (z, \widetilde{z})\,\mathbf{P}(E, \widetilde{E})\,(z', \widetilde{z}). \end{aligned}$$

Suppose the first relation of the claim is wrong, that is, $z\,\mathbf{R}(E)\,z'$, whereas $(z', \widetilde{z})\,\mathbf{P}(E, \widetilde{E})\,(z, \widetilde{z})$. A similar proof works for the second relation. By *Consistency*, $(z', \widetilde{z})\,\mathbf{P}(E, \widetilde{E})\,(z, \widetilde{z})$ implies that $z'\,\mathbf{P}(E)\,(z)$, the desired contradiction which proves the claim.

Back to the proof of the lemma. To simplify the notation, let money utility be measured by a function $u : \mathcal{R} \times (\mathcal{A}^* \times \mathbb{R}) \to \mathbb{R}$ defined by

$$u(R_i, z_i) = m \Leftrightarrow z_i\, I_i\, (\nu, m)\,.$$

Let us suppose that **R** satisfies *Independence of Preferences over Infeasible Bundles*, *Consistency*, and *Pareto Indifference*. Let $E = (N, A, R)$, $E' = (N, A, R') \in \mathcal{E}$, $z, z' \in Z(E)$, and $y, y' \in Z(E')$ be such that

$$u\left(R_i, z_i\right) = u\left(R'_i, y_i\right) \text{ and } u\left(R_i, z'_i\right) = u\left(R'_i, y'_i\right). \tag{1}$$

Let us assume that

$$z\,\mathbf{R}(E)\,z'. \tag{2}$$

We have to prove that $y\,\mathbf{R}(E')\,y'$. Let $n = \#N$. We begin by constructing two sets of n bundles which are infeasible for E. Let $\varepsilon > 0$. Let $\widetilde{m}, \widetilde{m}' \in \mathbb{R}^N$ be defined by

$$\widetilde{m}_i = \min\left\{-\varepsilon, u\left(R_i, z_i\right) - \varepsilon\right\}$$

$$\widetilde{m}'_i = \begin{cases} \widetilde{m}_i \text{ if } z_i \, I_i \, z'_i, \\ \widetilde{m}_i - \varepsilon \text{ if } z_i \, P_i \, z'_i, \\ \widetilde{m}_i + \varepsilon \text{ if } z'_i \, P_i \, z_i. \end{cases}$$

Let $\widetilde{A} \subset \mathcal{A}$ be such that $\widetilde{A} \cap (A \cup A') = \emptyset$ and $\#\widetilde{A} = n$, so that we can find a bijection $\sigma : N \to \widetilde{A}$. Let $\widetilde{N} \subset \mathcal{N}$ be such that $\widetilde{N} \cap N = \emptyset$ and $\#\widetilde{N} = n$, so that we can find a bijection $\rho : N \to \widetilde{N}$. Let $\widetilde{z}, \widetilde{z}' \in \left(\widetilde{A} \times \mathbb{R}\right)^{\widetilde{N}}$ be defined by: for all $i \in N$,

$$\begin{aligned} \widetilde{z}_{\rho(i)} &= (\sigma(i), \widetilde{m}_i), \\ \widetilde{z}'_{\rho(i)} &= (\sigma(i), \widetilde{m}'_i). \end{aligned}$$

Let $\overline{R} \in R^N$ be such that for all $i \in N, a, a' \in A \cup \{\nu\}, m, m' \in \mathbb{R}$,

$$(a, m) \; R_i \; (a', m') \Leftrightarrow (a, m) \; \overline{R}_i \; (a', m'),$$

and for all $i \in N$,

$$z_i \, \overline{I}_i \, \widetilde{z}_{\rho(i)} \text{ and } z'_i \, \overline{I}_i \, \widetilde{z}'_{\rho(i)}. \tag{3}$$

By the way $\widetilde{z}$ and $\widetilde{z}'$ were constructed, such preferences exist. By *Independence of Preferences over Infeasible Bundles*, Eq (2) implies

$$z \, \mathbf{R} \left(N, A, \overline{R}\right) \; z'. \tag{4}$$

Let $\widetilde{E} = \left(\widetilde{N}, \widetilde{A}, \widetilde{R}\right) \in \mathcal{E}$ with $\widetilde{R} \in \mathcal{R}^{\widetilde{N}}$ being defined by: for all $i \in N$,

$$\widetilde{R}_{\rho(i)} = \overline{R}_i. \tag{5}$$

By *Consistency* and the claim above, Eq (4) implies

$$(z, \widetilde{z}) \; \mathbf{R} \left(N \cup \widetilde{N}, A \cup \widetilde{A}, \left(\overline{R}, \widetilde{R}\right)\right) \; (z', \widetilde{z}).$$

By *Pareto Indifference*, and Eqs (3) and (5),

$$(\widetilde{z}, z) \; \mathbf{R} \left(N \cup \widetilde{N}, A \cup \widetilde{A}, \left(\overline{R}, \widetilde{R}\right)\right) \; (\widetilde{z}', z).$$

By *Consistency*,

$$\widetilde{z} \, \mathbf{R} \left(N, \widetilde{A}, \overline{R}\right) \; \widetilde{z}'. \tag{6}$$

Let $\overline{\overline{R}} \in \mathcal{R}^N$ be the list of preferences which coincides with $\overline{R}$ on the bundles having an element of $\widetilde{A} \cup \{\nu\}$ as first component, and with R' on the bundles having an element of $A' \cup \{\nu\}$ as first component, that is, for all $i \in N, a, a' \in \widetilde{A} \cup \{\nu\}, b, b' \in A' \cup \{\nu\}, m, m' \in \mathbb{R}$,

$$\begin{aligned} (a,m)\ \overline{\overline{R}}_i\ (a',m') &\Leftrightarrow (a,m)\ \overline{R}_i\ (a',m'),\ \text{and} \\ (b,m)\ \overline{\overline{R}}_i\ (b',m') &\Leftrightarrow (b,m)\ R'_i\ (b',m'). \end{aligned} \tag{7}$$

By *Independence of Preferences over Infeasible Bundles*, Eq (6) implies

$$\widetilde{z}\,\mathbf{R}\left(N, \widetilde{A}, \overline{\overline{R}}\right)\ \widetilde{z}'. \tag{8}$$

Let $\widetilde{\widetilde{R}} \in \mathcal{R}^{\tilde{N}}$ be defined by for all $i \in N$,

$$\widetilde{\widetilde{R}}_{\rho(i)} = \overline{\overline{R}}_i. \tag{9}$$

By *Consistency* and the claim above, Eq (8) implies

$$(\widetilde{z}, y)\ \mathbf{R}\left(N \cup \widetilde{N}, A' \cup \widetilde{A}, \left(\overline{\overline{R}}, \widetilde{\widetilde{R}}\right)\right)\ (\widetilde{z}', y).$$

By *Pareto Indifference* and Eqs (1), (3), (5) and (9),

$$(y, \widetilde{z})\ \mathbf{R}\left(N \cup \widetilde{N}, A' \cup \widetilde{A}, \left(\overline{\overline{R}}, \widetilde{\widetilde{R}}\right)\right)\ (y', \widetilde{z}).$$

By *Consistency*,

$$y\,\mathbf{R}\left(N, A', \overline{\overline{R}}\right)\ y'.$$

By *Independence of Preferences over Infeasible Bundles* and Eq (7),

$$y\,\mathbf{R}\,(E')\ y',$$

the desired outcome. ■

Before introducing fairness properties into the anlysis, let us discuss the result presented in Lemma 1. Welfarism, as is well-known, is the view that individual utilities are all what matters for equitable social decision making. Resource allocation decisions should all be reached by considering impacts on individual utilities. Once impacts on utilities are determined, choices are made on ground of an aggregation rule of individual utilities. This aggregation rule does not depend on the specific choice to be made, nor on the specific utility function of the agents (the literature has extensively studied the plausible aggregation rules; see d'Aspremont and Gevers [5]). On the other hand, it does not say anything about how (cardinally measurable and comparable) individual utilities shoulc be constructed.

It turns out that the *Money Utility Property* is equivalent to welfarism. But, moreover and more importantly, it also tells us how to construct utility representations of the preferences, as we are left with no choice but to aggregate money utilities, that is, the quantities of money which leave agents indifferent between consuming their assigned bundles or consuming the null object and receiving those quantities of money. Formally, the *money utility function* $u : \mathcal{R} \times (\mathcal{A}^* \times \mathbb{R}) \to \mathbb{R}$ is defined by $u(R_i, z_i) = m \Leftrightarrow z_i \, I_i \, (\nu, m)$.

Money Utility Welfarism For all $N \in \mathcal{N}$, there exists an ordering $\mathbf{R}^N$ on $\mathbb{R}^N$ such that for all $A \subset \mathcal{A}$, $R \in \mathcal{R}^N$ such that $E = (N, A, R) \in \mathcal{E}$, for all $z, z' \in Z(E)$,

$$z \, \mathbf{R}(E) \, z' \Leftrightarrow u \, \mathbf{R}^N \, u'$$

where $u_i = u(R_i, z_i)$ and $u'_i = u(R_i, z'_i)$ for all $i \in N$.

Lemma 2 *A social ordering function* $\mathbf{R}$ *satisfies the* Money Utility Property *if and only if it satisfies* Money Utility Welfarism*.*

Proof. "if": Let $E = (N, A, R), E' = (N, A', R') \in \mathcal{E}$, $z, z' \in Z(E)$, and $y, y' \in Z(E')$. By *Money Utility Welfarism*, there exists an ordering $\mathbf{R}^N$ on $\mathbb{R}^N$ such that

$$z \, \mathbf{R}(E) \, z' \Leftrightarrow u \, \mathbf{R}^N \, u'$$

where $u_i = u(R_i, z_i)$ and $u'_i = u(R_i, z'_i)$ for all $i \in N$, and

$$y \, \mathbf{R}(E') \, y' \Leftrightarrow v \, \mathbf{R}^N \, v'$$

where $v_i = u(R'_i, y_i)$ and $v'_i = u(R'_i, y'_i)$ for all $i \in N$. If there exist $m^*, m^{*\prime} \in \mathbb{R}^N$ such that

$$z_i \, I_i \, (\nu, m^*_i) \, I'_i \, y_i \text{ and } z'_i \, I_i \, (\nu, m^{*\prime}_i) \, I'_i \, y'_i$$

then for all $i \in N$, $u_i = v_i$ and $u'_i = v'_i$. Gathering the above relations yields

$$z \, \mathbf{R}(E) \, z' \Leftrightarrow y \, \mathbf{R}(E') \, y'.$$

"only if": Let $\mathbf{R}$ satisfy the *Money Utility Property.* Let us fix $N \in \mathcal{N}$ throughout the proof, and let us call $\mathcal{E}^N$ the set of economies where the set of agents is N. Let $E \in \mathcal{E}^N$. Let $X(E) \subset \mathbb{R}^N$ be defined by:

$$x \in X(E) \Leftrightarrow \exists\, z \in Z(E) \text{ s.t. } x_i = u(R_i, z_i), \forall\, i \in N.$$

By our assumption on preferences, $X(E)$ is compact. For $x, x' \in X(E)$, we write $x\,\mathbf{R}^N\,x'$ if and only if there exist $z, z' \in Z(E)$ such that for all $i \in N$, $x_i = u(R_i, z_i)$, $x'_i = u(R_i, z'_i)$ and $z\,\mathbf{R}(E)\,z'$. By the *Money Utility Property*, $\mathbf{R}^N$ is an ordering on $X(E)$. Let $E' \in \mathcal{E}^N$ and let $X(E')$ be defined as above. Let $x, x' \in X(E) \cap X(E')$. Let $z, z' \in Z(E)$, and $y, y' \in Z(E')$ be such that for all $i \in N$, $u(R_i, z_i) = u(R'_i, y_i)$ and $u(R_i, z'_i) = u(R'_i, y'_i)$. By the *Money Utility Property*, $z\,\mathbf{R}(E)\,z' \Leftrightarrow y\,\mathbf{R}(E')\,y'$. This proves that $\mathbf{R}$ is welfarist in its ranking over all allocations having a money utility representation in $X(E)$. Now, E was chosen arbitrarily. So $\mathbf{R}$ is welfarist on $X(E)$ for all $E \in \mathcal{E}^N$. It remains to show that $\cup_{E \in \mathcal{E}^N} X(E) = \mathbb{R}^N$. Let $x \in \mathbb{R}^N$. We must construct $E \in \mathcal{E}^N$ such that $x \in X(E)$. For all $i \in N$, let $a_i \in \mathcal{A}$ be such that $a_i \neq a_j$ for all $j \neq i$, and let $R_i \in \mathcal{R}$ be such that

$$\begin{aligned} x_i &\geq 0 \Rightarrow u(R_i, (a_i, 0)) = x_i, \text{ and} \\ x_i &< 0 \Rightarrow u(R_i, (a_i, x_i)) = x_i. \end{aligned}$$

Let $A = \cup_{i \in N} a_i$. Let $E = (N, A, R)$. By construction, $E \in \mathcal{E}^N$. Let z be defined by: for all $i \in N$, $z_i = (a_i, \min\{0, x_i\})$. By construction, $z \in Z(E)$. Also, $u(R_i, z_i) = x_i$ for all $i \in N$. Therefore, $x \in X(E)$. ■

In conclusion, in this specific model of indivisible good assignment with money, combining *Independence of Preferences over Infeasible Bundles*, *Consistency* and *Pareto Indifference* forces us to be welfarist and to use money utility as the proper indicator of individual welfare. But there is no restriction yet on how to aggregate money utilities. We come to this question in the next Section.

4 Fairness and the money utility leximin

The fair allocation literature which focusses on allocation rules has proposed a long list of equity axioms (see Moulin and Thomson [14]). One of the basic axioms, often called Equal Treatment of Equals, requires that if two agents have the same preferences, then they be assigned the same bundle, or at least, bundles they deem equivalent. Here we consider two possible adaptation of this requirement to our current framework. The first one is borrowed from Fleurbaey [6]. It refers to the Pigou-Dalton principle of transfer. This principle is at the heart of inequality measurement theory. It requires that an income transfer from an agent to a poorer one reduce inequality, as long as the income ranking of those two agents remains unaffected. That principle clearly favors equality but is consistent with any degree of inequality aversion. The Fleurbaey generalization of this principle to multi-dimensional framework like our requires that if two agents have the same preferences and are assigned bundles which do not lie on the same indifference curve, then a money transfer from the agent having been assigned the bundle they both prefer to the other agent be viewed a strict social improvement, provided there is no reversal in the indifference curve ranking, that is, provided both agents still consider the final bundle assigned to the first agent at least as good as that assigned to the second one.[3]

Transfer Principle among Equals For all $E = (N, A, R) \in \mathcal{E}$, $z = ((a_i, m_i))_{i \in N}$, $z' = ((a'_i, m'_i))_{i \in N} \in Z(E)$, if there exist $j, k \in N$ such that $R_j = R_k$ and for all $i \neq j, k, z_i = z'_i$, then for all $\Delta > 0$,

$$\left[\left(a'_j, m'_j\right) = (a_j, m_j - \Delta)\ R_{j,k}\ (a'_k, m'_k) = (a_k, m_k + \Delta)\right] \Rightarrow [z' \, \mathbf{P}(E) \, z] \, .$$

Our second fairness axiom captures the anonymity content of the classical *Equal Treatment of Equals* axiom. It requires that if two agents have the same preferences, then permuting the two bundles they are assigned in an allocation yield an allocation the society deems equivalent. This requirement is also compatible with any degree of inequality aversion.[4]

Anonymity among Equals For all $E = (N, A, R) \in \mathcal{E}$, z, $z' \in Z(E)$, if there exist $j, k \in N$ such that $R_j = R_k$ and for all $i \neq j, k, z_i = z'_i$, then

$$\left[z_j = z'_k \text{ and } z'_j = z_k\right] \Rightarrow [z \, \mathbf{I}(E) \, z'] \, .$$

The main result of the paper, which we state and prove at the end of this Section, shows that the combination of these fairness axioms with our

[3]If the proviso that agents' preferences are identical is removed, and if no other restriction is imposed on the preferences of those two agents, then the resulting (much stronger) axiom turns out to be incompatible with *Pareto Indifference*. See Fleurbaey and Trannoy [9] for a proof and discussion.

[4]It is even compatible with a negative degree of inequality aversion.

other axioms leads us to be infinitely averse to money utility inequality. Infinite inequality aversion is captured by the so-called leximin, or lexicographic maximin, aggregation criterion. For $m, m' \in \mathbb{R}^N$, we write $m \geq_{\text{lex}} m'$ if and only if the smallest element of m is greater than the smallest element of m', or they are equal but the second smallest element of m is greater than the second smallest element of m', and so on.

Definition 1 *The* Money Utility Leximin *function* $\mathbf{R}^L$ *works as follows: For all* $E = (N, A, R) \in \mathcal{E}$*,* $z, z' \in Z(E)$*, let* $m^*, m^{*\prime} \in \mathbb{R}^N$ *be such that*

$$z_i\, I_i\, (\nu, m_i^*) \text{ and } z_i'\, I_i\, (\nu, m_i^{*\prime});$$

then

$$z\, \mathbf{R}(E)\, z' \Leftrightarrow m^* \geq_{lex} m^{*\prime}.$$

As we want to insist in this paper on positive results, we begin by proving that any combination of axioms defined so far leads to some possibility. The *Money Utility Leximin* function, indeed, satisfies all our axioms.

Lemma 3 *The* Money Utility Leximin *function* $\mathbf{R}^L$ *satisfies* Independence of Preferences over Infeasible Bundles*,* Consistency*,* Strong Pareto*, the* Transfer Principle among Equals *and* Anonymity among Equals*.*

Proof. To simplify the notation, we use the money utility function u introduced in Section 3. 1) *Independence of Preferences over Infeasible Bundles*: Let $E = (N, A, R)$, $E' = (N, A, R') \in \mathcal{E}$, be such that for all $i \in N, a, a' \in A \cup \{\nu\}, m, m' \in \mathbb{R}, (a, m)\ R_i\ (a', m') \Leftrightarrow (a, m)\ R_i'\ (a', m')$. Then for all $z, z' \in Z(E)$, and all $i \in N$, we have $u\,(R_i, z_i) = u\,(R_i', z_i)$ and $u\,(R_i, z_i') = u\,(R_i', z_i')$, so that $z\, \mathbf{R}^L\,(E)\ z' \Leftrightarrow z\, \mathbf{R}^L\,(E')\ z'$. 2) *Consistency*: This comes from the fact that for all $m, m' \in \mathbb{R}^N, M \subset N$, if $m_i = m_i'$ for all $i \in N \setminus M$, then $m \geq_{\text{lex}} m' \Rightarrow m_M \geq m_M'$ and $m >_{\text{lex}} m' \Rightarrow m_M > m_M'$.[5] 3) *Strong Pareto:* This comes from the fact that for all $m, m' \in \mathbb{R}^N$, if $m \geq m'$, then $m \geq_{\text{lex}} m'$, and, if, in addition, $m_j > m_j'$ for some $j \in N$, then $m >_{\text{lex}} m'$. 4) *Transfer Principle among Equals:* Let $E = (N, A, R) \in \mathcal{E}$, $z = ((a_i, m_i))_{i \in N}$, $z' = ((a_i', m_i'))_{i \in N} \in Z(E)$, be such that for some $j, k \in N$, $R_j = R_k$ and for all $i \neq j, k, z_i = z_i'$, and, moreover,

$$\left[\left(a_j', m_j'\right) = (a_j, m_j - \Delta)\ R_{j,k}\ (a_k', m_k') = (a_k, m_k + \Delta)\right].$$

[5]Our conventions for vector inequalities: for $x, y \in R^L, x \geq y \Leftrightarrow x_l \geq y_l, \forall\, l \in L$, and $x > y \Leftrightarrow x \geq y$ and $x \neq y$.

Then, $u(R_j,(a_j,m_j)) > u\left(R_j,\left(a'_j,m'_j\right)\right) \geq u(R_k,(a'_k,m'_k)) > u(R_k,(a_k,m_k))$, so that $z'\,\mathbf{P}^L(E)\,z$. 5) *Anonymity among Equals*: Let $E=(N,A,R)\in\mathcal{E}$, z, $z'\in Z(E)$, and $j,k\in N$ be such that $R_j=R_k$, for all $i\neq j,k, z_i=z'_i$, and $z_j=z'_k$ and $z'_j=z_k$. Then $u(R_j,z_j)=u(R_k,z'_k)$ and $u\left(R_j,z'_j\right)=u(R_k,z_k)$, so that $z\,\mathbf{I}^L(E)\,z'$. ■

Now, we pursue our analysis of money utility by showing that the fairness axioms defined above, when combined with *Strong Pareto* and the *Money Utility Property* force us to be infinitely averse to money utility inequality, that is, characterize the *Money Utility Leximin* function $\mathbf{R}^L$. Actually, the role played by each one of our fairness axioms turns out to be clear. *Transfer Principle among Equals* yields the maximin property in money utility, whereas *Anonymity among Equals* implies that the maximin be applied lexicographically. This is made clear through the following two lemmas.

Money Utility Maximin Property For all $E=(N,A,R)\in\mathcal{E}$, for all $z,z'\in Z(E)$, if $m^*,m^{*\prime}\in\mathbb{R}^N$ are such that $z_i\,I_i\,(\nu,m_i^*)$ and $z'_i\,I_i\,(\nu,m_i^{*\prime})$ then

$$\min_{i\in N}\{m_i^*\} > \min_{i\in N}\{m_i^{*\prime}\} \Rightarrow z\,\mathbf{P}(E)\,z'.$$

Lemma 4 *If a social ordering function* $\mathbf{R}$ *satisfies the* Money Utility Property, Strong Pareto *and the* Transfer Principle among Equals*, then it satisfies the* Money Utility Maximin Property *in all* $E=(N,A,R)\in\mathcal{E}$ *such that* $\#A\geq 2$.

Proof. Again, we use the u function terminology. Let $E=(N,A,R)\in\mathcal{E}$, $z,z'\in Z(E)$, be such that

$$\min_{i\in N}\{u(R_i,z_i)\} > \min_{i\in N}\{u(R_i,z'_i)\}.$$

Assume, contrary to what need to be proven, that $z'\,\mathbf{R}(E)\,z$. Let $E'=(N,A,R')\in\mathcal{E}$ be such that for all $j,k\in N$, $R'_j=R'_k$ and R'_j has the property that for all $a\in A$, $m,m'\in\mathbb{R}$, and for all $i\in N$,

$$(a,m)\ I_i\,(\nu,m') \Rightarrow (a,m)\ R'_j\,(\nu,m'),$$

which means that the willingness to pay for any object in A is greater for R'_j than for any R_i. Moreover, we also assume that R'_j satisfies condition C which is defined below. Given the restriction on R'_j, there exist $y,y'\in Z(E')$ such that

$$u(R_i,z_i)=u(R'_i,y_i)\ \text{ and } u(R_i,z'_i)=u(R'_i,y'_i). \tag{10}$$

By the *Money Utility Property*, Eq (10) implies

$$y' \, \mathbf{R}(E') \, y. \tag{11}$$

Let $j \in N$ be such that $u\left(R'_j, y'_j\right) = \min_{i \in N} u\left(R'_i, y'_i\right)$. Let N be partitionned into N_1, and N_2 such that $\#N_1 = n_1, \#N_2 = n_2$ and

$$\begin{aligned} \forall\, i &\in N_1 : u(R'_i, y_i) \geq u(R'_i, y'_i), \\ \forall\, i &\in N_2 : u(R'_i, y_i) < u(R'_i, y'_i). \end{aligned}$$

Note that $j \in N_1$. If $N_2 = \emptyset$, then, by *Strong Pareto*, $y \, \mathbf{R}(E') \, y'$, a contradiction. So, let us assume that $N_2 \neq \emptyset$. The remaining of the proof consists in showing that it is possible to build a new allocation y'', such that $y'' \, \mathbf{P}(E') \, y$ and N can still be partitionned into two sets, N'_1 and N'_2, such that $\#N'_1 = n_1 + 1$ and $\#N'_2 = n_2 - 1$. Repeating the argument n_2 times eventually yield the contradiction with *Strong Pareto*. Note that each repetition of the argument typically requires that new preferences be defined, which, by the *Money Utility Property*, is always possible. Let $k \in N_2$. Let u''_j, u''_k be such that

$$u\left(R'_j, y'_j\right) < u''_j \leq u''_k < u\left(R'_l, y_l\right), \forall\, l = j, k.$$

Let $a, b \in A$. We may and do assume that $a \neq b$, since $\#A \geq 2$. We are now ready to define condition C. There is some $\Delta > 0$, and $m_j, m_k \in \mathbf{R}$, such that

$$\begin{aligned} & y'_j \, I'_j \, \left(a, m_j\right), \\ & \left(\nu, u''_j\right) \, I'_j \, \left(a, m_j + \Delta\right), \\ & \left(\nu, u''_k\right) \, I'_k \, \left(b, m_k - \Delta\right), \\ & y'_k \, I'_k \, \left(b, m_k\right). \end{aligned}$$

Let $\overline{y}, y'' \in Z(E')$ be such that for all $i \neq j, k$, $\overline{y}_i = y''_i \, I'_i \, y'_i$, $\overline{y}_j = (a, m_j)$, $\overline{y}_k = (b, m_k)$, $y''_j = (a, m_j + \Delta)$, and $y''_k = (b, m_k - \Delta)$. By *Pareto Indifference*,

$$y' \, \mathbf{I}(E') \, \overline{y}. \tag{12}$$

By the *Transfer Principle among Equals*,

$$y'' \, \mathbf{P}(E') \, \overline{y}. \tag{13}$$

By Eqs (11), (12) and (13),

$$y'' \, \mathbf{P}(E') \, y,$$

the desired outcome. ■

There is an interesting relationship between this lemma and results in Fleurbaey [6] and Maniquet and Sprumont [12]. In those two papers, indeed, fairness axioms a priori consistent with any degree of inequality aversion also yield maximin properties when they are combined with, typically, independence axioms. This common feature of the domains where social ordering functions have been studied up to now stands in contrast to the income inequality measurement framework as well as the social welfare functionals framework. In the former model, the Pigou-Dalton principle of transfer, even when combined with different types of independence and consistency properties do not yield the maximin property (see Chakravarty [4]). In the latter setting, the traditional equity axiom which leads to the maximin axiom, that is, the Hammond equity axiom, excludes by itself any finite degree of utility inequality aversion (cfr. Hammond [10]; see also d'Aspremont and Gevers [5] for a survey).

Lemma 5 *If a social ordering function satisfies the* Money Utility Property, *the* Maximin Money Utility Property, Consistency *and* Anonymity among Equals, *then it coincides with the* Money Utility Leximin *function* $\mathbf{R}^L$ *in all* $E = (N, A, R) \in \mathcal{E}$ *such that* $\#A \geq 2$.

Proof. First let us note that the *Money Utility Property* clearly implies *Pareto Indifference*. Let $\mathbf{R}$ satisfy the axioms. Let $E = (N, A, R) \in \mathcal{E}$ be such that $\#A \geq 2$. By the *Money Utility Property*, we can assume, w.l.o.g., that for all $i, j \in N, R_i = R_j$. (If this condition were not satisfied, then we could change the actual profile of preferences into a new profile satisfying the condition, without affecting the social preferences over money utility vectors, as we did at the beginning of the proof of Lemma 4.) Let $z = ((a_i, m_i))_{i \in N}, z' = ((a'_i, m'_i))_{i \in N} \in Z(E)$. We distinguish two cases.
Case 1: $z \, \mathbf{I}^L(E) \, z'$. Let $n = \#N$. Let $\sigma : N \to \{1, \ldots, n\}$ be a bijection satisfying the property that for all $i, j \in N$, $\sigma(i) \geq \sigma(j) \Rightarrow u(R_i, z_i) \geq u(R_j, z_j)$. Let $z_\sigma \in Z(E)$ denote the allocation obtained from z by permuting its component according to σ. Let σ' denote the similar bijection associated to z', and $z'_{\sigma'}$ the resulting allocation. By *Anonymity among Equals*, $z \, \mathbf{I}(E) \, z_\sigma$ and $z' \, \mathbf{I}(E) \, z'_{\sigma'}$. By construction, and given that $z \, \mathbf{I}^L(E) \, z'$, $z_{\sigma i} \, I_i \, z'_{\sigma' i}$ so that by

Pareto Indifference, $z_\sigma \, \mathbf{I}(E) \, z'_{\sigma'}$. Gathering all the above social indifferences yield $z \, \mathbf{I}(E) \, z'$, the desired outcome.
Case 2: $z \, \mathbf{P}^L(E) \, z'$. Assume, contrary to the statement we have to prove, that

$$z' \, \mathbf{R}(E) \, z. \tag{14}$$

Let $n = \#N$. Let $\sigma, \sigma', z_\sigma, z'_{\sigma'}$ be defined as above. By *Anonymity among Equals*, $z \, \mathbf{I}(E) \, z_\sigma$ and $z' \, \mathbf{I}(E) \, z'_{\sigma'}$.Therefore, by Eq (14),

$$z'_{\sigma'} \, \mathbf{R}(E) \, z_\sigma. \tag{15}$$

Given that $z \, \mathbf{P}^L(E) \, z'$, there is $j \in N$ such that for all $i \in N$ such that $\sigma(i) < \sigma(j), z_{\sigma i} \, I_i \, z'_{\sigma' i}$ and $z_{\sigma j} \, P_j \, z'_{\sigma' j}$. Then, N can be partitionned into N_1, N_2 and N_3 such that $\#N_1 = n_1, \#N_2 = n_2, \#N_3 = n_3$ and $j = n_1 + 1$, and

$$\begin{aligned} \forall i &\in N_1 : \sigma(i) \leq n_1 \text{ and } u(R_i, z_{\sigma i}) = u(R_i, z'_{\sigma' i}), \\ \forall i &\in N_2 : u(R_i, z_{\sigma i}) > u(R_i, z'_{\sigma' i}), \\ \forall i &\in N_3 : u(R_i, z_{\sigma i}) \leq u(R_i, z'_{\sigma' i}). \end{aligned}$$

If $N_1 = \emptyset$, then by the *Maximin Money Utility Property*, $z_\sigma \, \mathbf{P}(E) \, z'_{\sigma'}$, a contradiction. So let us assume that $N_1 \neq \emptyset$. Our strategy consists in using *Consistency* to remove agents in N_1 from the economy, so that agent j has the smallest money utility, in contradiction to the *Maximin Money Utility Property*, but removing those agents may yield an infeasible allocation (not enough money would be left). Let $M = \min \left\{ \sum_{i \in N_1} m_{\sigma i}, \sum_{i \in N_1} m'_{\sigma' i} \right\}$. Let $\widetilde{N} \subset \mathcal{N}, \widetilde{A} \subset \mathcal{A}$ be such that $\widetilde{N} \cap N = \emptyset, \#\widetilde{N} = n$, $\widetilde{A} \cap A = \emptyset$ and $\#\widetilde{A} = n$. Let $\widetilde{z} = ((\widetilde{a}_i, \widetilde{m}_i))_{i \in \widetilde{N}} \in (\mathcal{A}^* \times \mathbb{R})^{\widetilde{N}}$, be such that $\sum_{i \in \widetilde{N}} \widetilde{m}_i < M$. Let $\widetilde{E} = \left(\widetilde{N}, \widetilde{A}, \widetilde{R} \right) \in E$ be such that for all $i \in \widetilde{N}$, $\widetilde{R}_i = R_j$. By the *Money Utility Property*, we can assume, w.l.o.g., that for all $i \in \widetilde{N}$,

$$u\left(\widetilde{R}_i, \widetilde{z}_i\right) > u\left(R_j, z'_{\sigma' j}\right).$$

(If this condition were not satisfied, then we could change the actual profile of preferences into a new profile satisfying the condition, without affecting the social preferences over money utility vectors.) Let $\overline{z}, \overline{z}' \in Z(E)$ be defined by

$$\begin{aligned}\forall i \in N_2 \cup N_3, \overline{z}_i &= z_{\sigma i} \text{ and } \overline{z}'_i = z'_{\sigma' i}, \\ \forall i \in N_1, \overline{z}_i = \overline{z}'_i &= z_{\sigma i} \text{ if } \sum_{i\in N_1} m_{\sigma i} \leq \sum_{i \in N_1} m'_{\sigma' i}, \\ & \quad z'_{\sigma' i} \text{ if } \sum_{i\in N_1} m_{\sigma i} > \sum_{i \in N_1} m'_{\sigma' i}.\end{aligned}$$

By *Pareto Indifference*, $z_\sigma \, \mathbf{I}(E) \, \overline{z}$ and $z'_{\sigma'} \, \mathbf{I}(E) \, \overline{z}'$, so that Eq (15) implies

$$\overline{z}' \, \mathbf{R}(E) \, \overline{z}.$$

By *Consistency* and the claim at the beginning of the proof of Lemma 1,

$$(\overline{z}', \widetilde{z}) \; \mathbf{R}(E, \widetilde{E}) \; (\overline{z}, \widetilde{z}).$$

By *Consistency*,

$$\left(\overline{z}'_{N \setminus N_1}, \widetilde{z}\right) \mathbf{R} \left((N \setminus N_1) \cup \widetilde{N}, (A \setminus \cup_{i \in N_1} \{\overline{a}_i\}) \cup \widetilde{A}, \left(R_{N \setminus N_1}, \widetilde{R}\right)\right) \; \left(\overline{z}_{N \setminus N_1}, \widetilde{z}\right),$$

which contradicts the *Maximin Money Utility Property*, as agent j now has the smallest money utility in the allocation $\left(\overline{z}'_{N \setminus N_1}, \widetilde{z}\right)$. ■

We can gather the results presented so far and state the following theorem.

Theorem 1 *A social ordering function* $\mathbf{R}$ *satisfies* Independence of Preferences over Infeasible Bundles*,* Consistency*,* Strong Pareto*, the* Transfer Principle for Equals*, and* Anonymity among Equals *if and only if it is the* Money Utility Leximin *function* $\mathbf{R}^L$.

Proof. The "if" part was proven in Lemma 3. The "only if" was proven in Lemmas 1, 4 and 5 in the case of economies $E = (N, A, R) \in \mathcal{E}$ such that $\#A \geq 2$. We only need to get rid of this last restriction. But by *Consistency* and the Claim at the beginning of the proof of Lemma 1, we can always go from one object economies to two or more objects economies. Lemma 4 is therefore valid on the entire domain $\mathcal{E}$, and so is Lemma 5. ■

Let us check now that none of the above axioms is redundant.

Let $a^* \in \mathcal{A}$. Let numerical representation of preferences v be defined by

$$v(R_i, z_i) = m \Leftrightarrow (a^*, m) \; I_i \, z_i.$$

The v-utility leximin function satisfies all the axioms but *Independence of Preferences over Infeasible Bundles.*

For $a \in \mathcal{A}^*$, let $m^a(R_i, z_i) = m \Leftrightarrow (a, m)\ I_i\ z_i$. For all $E = (N, A, R) \in \mathcal{E}$, let the numerical representation of preferences w be defined by

$$w(R_i, z_i, A) = m \Leftrightarrow m = \sum_{a \in A \cup \{\nu\}} m^a(R_i, z_i).$$

The w-utility leximin function satisfies all the axioms but *Consistency.*

The social ordering function which ranks allocations so as to minimize the variance in money utilities satisfies all the axioms but *Strong Pareto.*

The social ordering function which ranks allocations so as to maximize the sum of agents' money utility satisfies all the axioms but the *Transfer Principle for Equals.*

Let $\geq$ denote a complete ordering on $\mathcal{N}$. The social ordering function which coincides with the money utility leximin function in case of strict preference, and which prefers, in case of a tie by the leximin, the allocation where the name of agent with smallest money utility is the smaller, satisfies all the axioms but *Anonymity among Equals.*

5 Conclusion

The particular feature of the indivisible good assignment model that is exploited in this paper is that it is always feasible not to assign any indivisible good to one agent (that is, assign her only money). This comes from the assumption that we can freely dispose of the objects. Consequently, a share of agents' consumption set always contains feasible bundles and, moreover, is the intersection of all possible feasible sets. This is why the information about which quantity of money would leave an agent indifferent between consuming her actual bundle and not consuming any object but receiving this amount of money is crucial. As it turns out, this information is sufficient to construct social ordering functions.

If we had assumed, however, that there was no free disposal, so that a social ordering only has to rank the allocations where all available objects are allocated, then all the results we obtained would still be valid, provided the number of objects were assumed to be strictly lower than the number of agents.[6] Under this assumption, indeed, it is always the case that at least one agent does not consume any object, but this agent may be anyone, so that the set of feasible bundles always include the null object bundles.

Let us conclude by drawing two general lessons from this paper. First of all, our result (together with results in the other recent papers studying fair social orderings) shows that fairness can be studied through the definition of social orderings and not only allocation rules. As explained above, social orderings offer much finer judgements than allocation rules.[7] On the other hand, allocation rules can easily be deduced from social ordering functions, by simply taking the selection to be the set of socially preferred allocations. The allocation rule associated with the money utility leximin function then selects the subset of efficient allocations where all agents have the same money utility, that is, they all are indifferent between their assigned bundle and a common null object bundle. Let us call it the *Equal Money Utility* rule. It is an egalitarian-equivalent allocation rule, a concept introduced by Pazner and Schmeidler [15].

There is no clear logical relationship, however, between the analysis of social ordering functions and the analysis of allocation rules. The model studied in this paper offers the best example of this fact. For the *Equal Money Utility* rule does not rely on preferences over infeasible bundles, is consistent, selects all Pareto indifferent allocations, is Pareto efficient and treats equals equally. In view of Theorem 1, one may therefore wonder if it is the only rule satisfying those requirements. It is not, as proven by the following example. The allocation rule which has, by far, received the largest attention in the literature, is the one selecting the efficient and envy-free allocations (see Alkan [1], Alkan, Demange and Gale [2], Maskin [13], Svensson [18], and Tadenuma and Thomson [19]; an exception is Bevia [3]). An allocation is envy-free if no agent strictly prefers the bundle assigned to any other agent over her own bundle. The efficient and envy-free rule is independent of preferences over infeasible bundles, consistent, and efficient. It treats equals equally, and selects Pareto indifferent allocations. But that

[6] This would also require rewriting the *Consistency* property, and adapting the way it is used in the proofs.

[7] It is often argued that the success of the theory of fair allocation (in the sense of it uncovering mainly possibility rather than impossibility results) comes from the fact that the attention is put on allocation rules rather than orderings (see e.g. Varian [23], Sen [17] and Moulin and Thomson [14]). It is clear after this paper and the other papers on social ordering functions that the argument does not hold true.

rule is disjoint from the *Equal Money Utility* rule (this is a corollary of the main result of Thomson [20]).

Our second general lesson is the following. This paper shows that, contrary to the general wisdom which is currently dominant in welfare economics, it is possible to build social orderings in economic domains without relying on interpersonal comparison of utility, so that, in particular, the social planner objective in public economics need not be utilitarian. At least, this has been shown in a few models. An urgent task in welfare economics, therefore, is to extend this line of research to other models as well.

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